Lecture 3

Manifolds

Recap

Where were we?

Maxwell Equations

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= -J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

$$\partial_{[\alpha} F_{\mu\nu]} = 0$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

Lorenz gauge

$$\partial_\mu A^\mu = 0$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = \overbrace{\partial_\mu \partial^\mu}^{\overset{\text{D'Alembertian}}{\Box}} A^\nu - \partial^\nu \overbrace{(\partial_\mu A^\mu)}^{\overset{0}{\shortparallel}}$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu = -J^\nu$$

Maxwell Equations

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= -J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

Lorenz gauge

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \quad \partial_\mu \partial^\mu A^\nu = -J^\nu$$

Lorentz force :

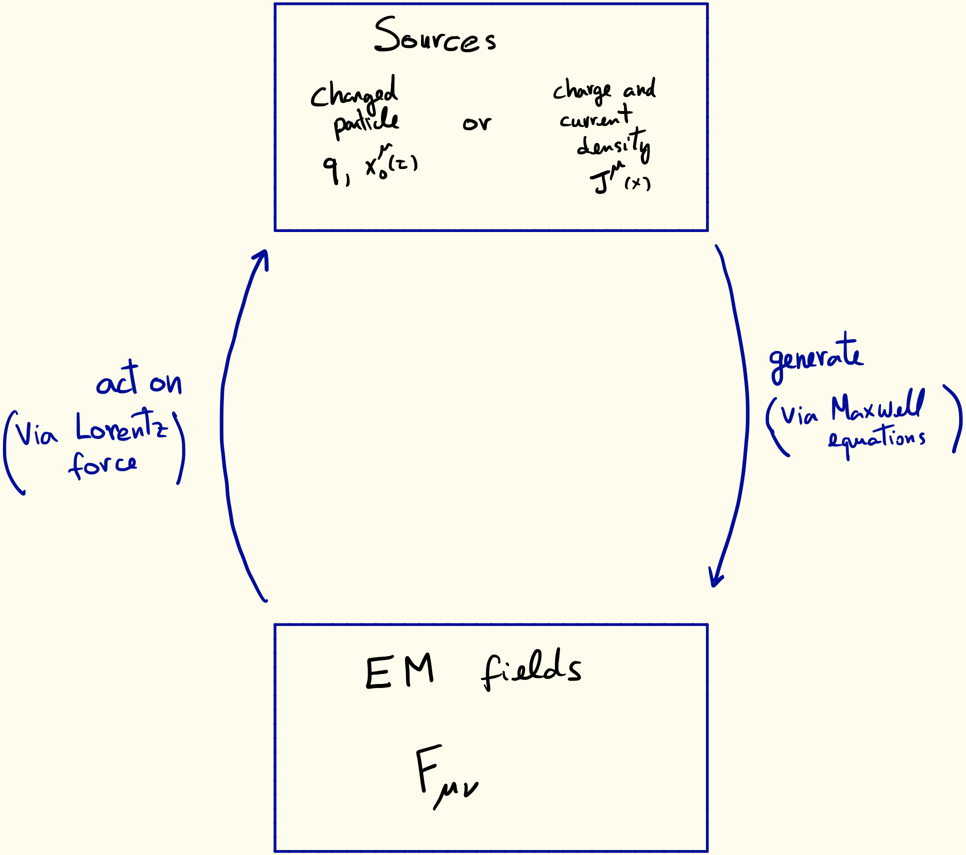
$$\boxed{\frac{dp^\mu}{d\tau} = q\, F^{\mu\nu} U_\nu}$$

$$p^\mu = m\, U^\mu = \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix}$$

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

proper Time

$$p^\mu p_\mu = m^2 U^\mu U_\mu = -m^2$$


Sources
Charged particle
q, $x_0^\mu(\tau)$
or
charge and current density
$J^\mu(x)$
generate
(via Maxwell equations)
act on
(via Lorentz force)
EM fields
$F_{\mu\nu}$

Current 4-vector of a charged particle

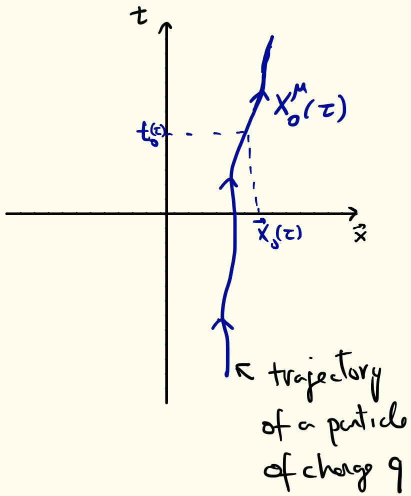
$$J^\mu(x) = \ ?$$

$$J^\mu(\underset{\substack{\uparrow \\ (t,\vec{x})}}{x}) = q \int d\tau \, \underbrace{\delta^{(4)}\big(x - x_0(\tau)\big)}_{\delta(t - t_0(\tau))\, \delta^{(3)}(\vec{x} - \vec{x}_0(\tau))} \frac{dX_0^\mu(\tau)}{d\tau}$$

$$= q \int \frac{dt_0}{\gamma} \, \delta(t - t_0) \, \delta^{(3)}\big(\vec{x} - \underbrace{\vec{x}_0(\tau(t_0))}_{\substack{\equiv \\ \vec{x}_0(t_0) \text{ notation}}}\big) \, u^\mu(t_0)$$

$$= q \, \frac{u^\mu(t)}{\gamma} \, \delta^{(3)}\big(\vec{x} - \vec{x}_0(t)\big)$$

$$= q \begin{pmatrix} 1 \\ \vec{v}(t) \end{pmatrix} \delta^{(3)}\big(\vec{x} - \vec{x}_0(t)\big)$$



Energy-momentum Tensor

$$[T^{\mu\nu}] = \frac{[p^\mu]}{L^3} \leftarrow \prod_{\substack{\alpha=0 \\ (\alpha\neq\nu)}}^{3} \Delta x^\alpha$$

$T^{\mu\nu}$ = flux (per unit area) of momentum p^μ across a surface of constant x^ν

✓ symmetric : $T^{\mu\nu} = T^{\nu\mu}$ ← can be made

✓✓ conserved : $\partial_\nu T^{\mu\nu} = 0$

GR definition

Noether's theorem (QFT)

For a perfect fluid in its rest frame : $[U^\mu = (1,0,0,0)]$

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$\rho = T^{00}$ = energy density

p = pressure

In a general frame, we can write

4-velocity field of the fluid $U^\mu = \gamma(1, \vec{v})$

$$T^{\mu\nu} = (\rho + p)\, U^\mu U^\nu + p\, \eta^{\mu\nu}$$

Energy-momentum Tensor

$$\boxed{T^{\mu\nu} = (\rho + p)\, U^\mu U^\nu + p\, \eta^{\mu\nu}}$$

4-velocity field of the fluid

$$U^\mu = \gamma\,(1, \vec{v})$$

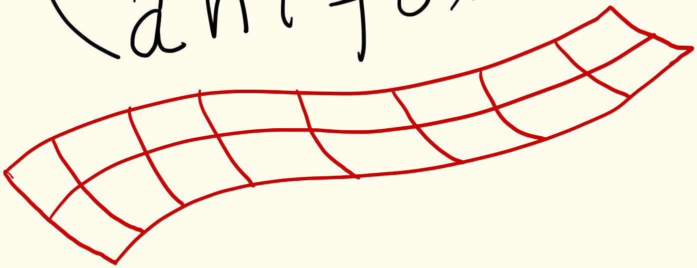
Exercise: Compute $\partial_\mu T^{\mu\nu}$ in the non-relativistic limit: $|\vec{v}| \ll 1$, $p \ll \rho$.

Show that:

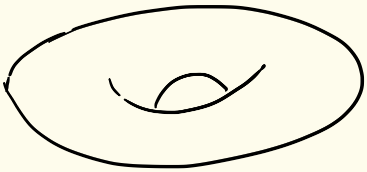
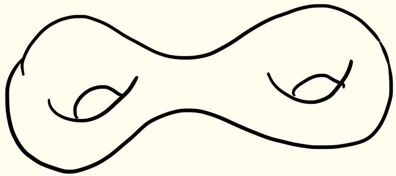
$$\partial_\mu T^{\mu 0} = 0 \iff \partial_t \rho + \vec{\nabla}\cdot(\rho \vec{v}) = 0 \qquad \text{Continuity equation}$$

$$\partial_\mu T^{\mu i} = 0 \iff \rho\left[\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\,\vec{v}\right] = -\vec{\nabla} p \qquad \text{Euler equation}$$

Manifolds



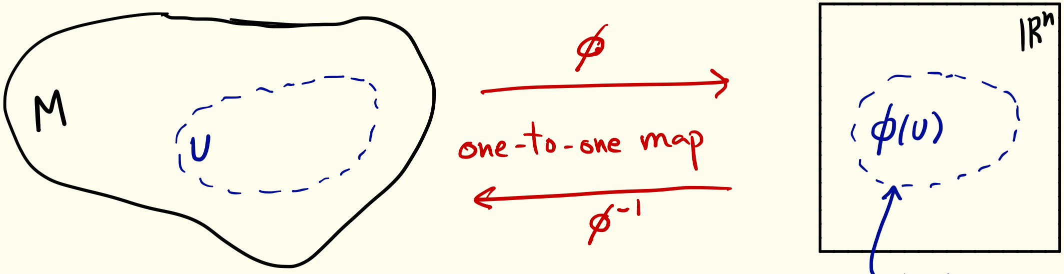
Manifolds

Intuitively, a manifold is a space that locally looks like $\mathbb{R}^n$

↳ dimension $= n$

Chart or **coordinate system**: $\phi : U \longrightarrow \mathbb{R}^n$



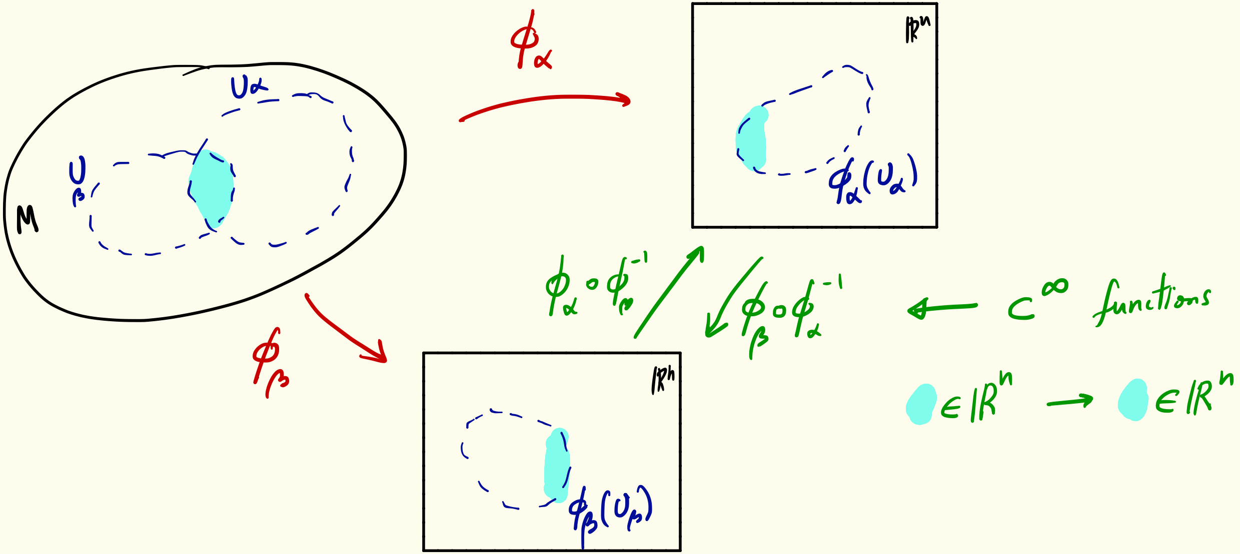
$B_{y,r} = \{ x \in \mathbb{R}^n \mid |x-y| < r \}$

open set (union of open ball)

A C^∞ atlas is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that:

- The atlas covers M. $\iff$ $\bigcup_\alpha U_\alpha = M$ (Union)

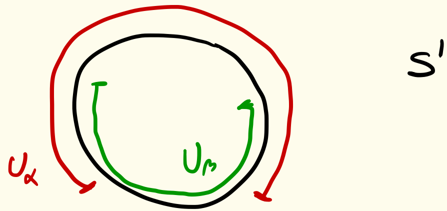
- The charts are smoothly (C^∞) sewn together



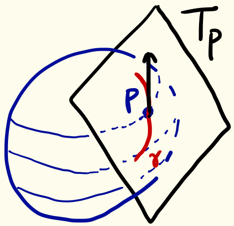
A C^∞ n-dimensional manifold is a set M along with a maximal atlas, one that contains every possible compatible chart.

Comments:

- $C^\infty \to C^p$
- No need for embedding M in $\mathbb{R}^m$ with $m > n$.
- In general one chart is not enough

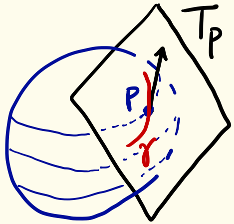


Tangent space



T_p = "space of tangent vectors to curves"

Tangent space

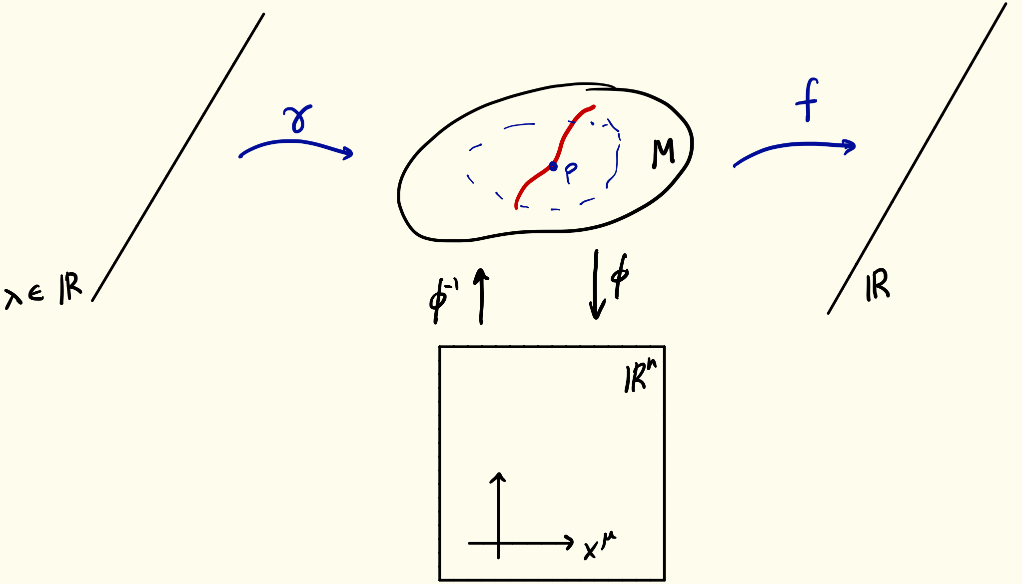


T_p = "space of tangent vectors to curves"

Given a C^∞ function $f : M \longrightarrow \mathbb{R}$ we can define its **directional derivative** at p :

$$\frac{d}{d\lambda} f \equiv \frac{d}{d\lambda} f(\gamma(\lambda))\Big|_{\gamma(\lambda)=p}$$

T_p = space of directional derivatives at p.



$$\frac{d}{d\lambda} f = \frac{d}{d\lambda}(f \circ \gamma) = \frac{d}{d\lambda}\Big(\underbrace{f \circ \phi^{-1} \circ \underbrace{\phi \circ \gamma}_{x^\mu(\lambda)}}_{f(x^\mu(\lambda))}\Big) = \frac{dx^\mu}{d\lambda}\, \partial_\mu f \qquad \forall f$$

$$\Rightarrow \quad \boxed{\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\, \partial_\mu}$$

coordinate basis of T_p

Change of coordinates $x^{\mu} \longrightarrow x^{\mu'}$

$\Rightarrow$ Change of basis $\partial_{\mu'} = \dfrac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$

Vector $V = V^{\mu} \partial_{\mu}$ should remain unchanged:

$$V = \underbrace{V^{\mu}}_{\text{components}} \underbrace{\partial_{\mu}}_{\text{basis elements}} = V^{\mu'} \partial_{\mu'} = V^{\mu'} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$$

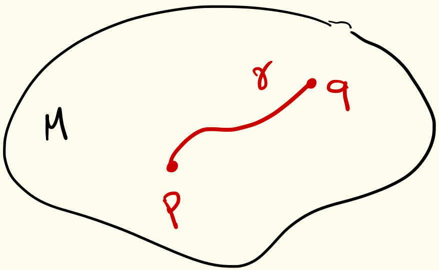
$$\Rightarrow \boxed{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} V^{\nu}}$$ Vector transformation law

$$\frac{\partial x^{\nu'}}{\partial x^{\mu}} V^{\mu} = V^{\mu'} \underbrace{\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\mu}}}_{\delta^{\nu'}_{\mu'}}$$

The metric is a $(0,2)$ symmetric tensor that determines the line element:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

metric components



$\gamma(0) = p$
$\gamma(1) = q$

$$\text{Length} = \int_0^1 d\lambda \sqrt{g_{\mu\nu}(x(\lambda))\, \dot{x}^\mu\, \dot{x}^\nu}\,, \quad \bullet \equiv \frac{d}{d\lambda}$$

$$\text{Length} = \int_p^q ds$$

Wisdom of the day

You are not your thoughts.

Read more at: www.mindfulness.swiss

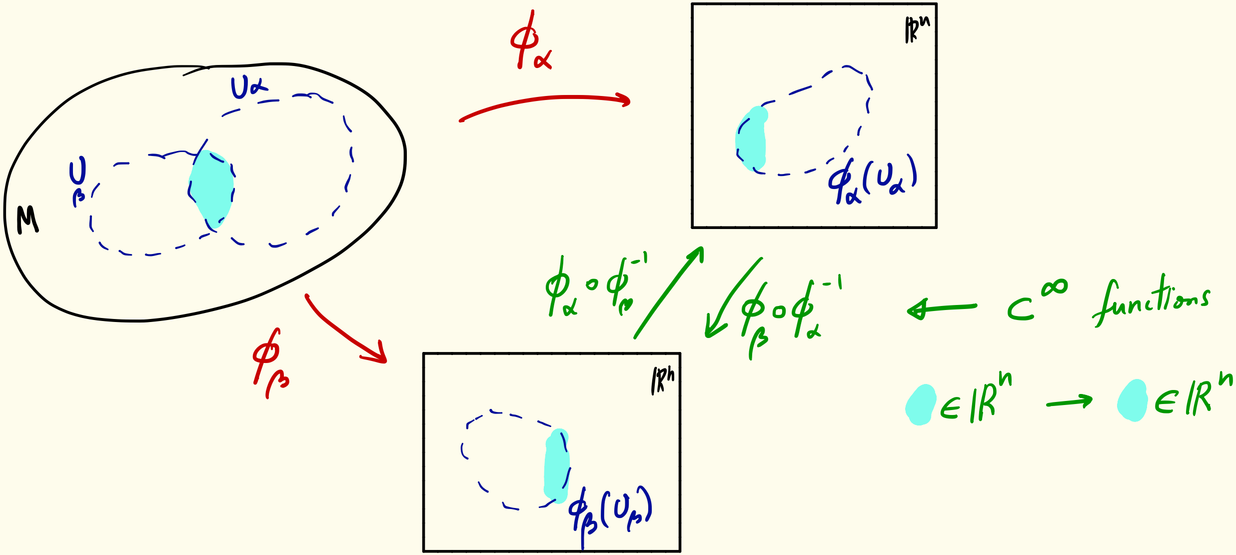
Lecture 4

Tensors and differential forms

Recap

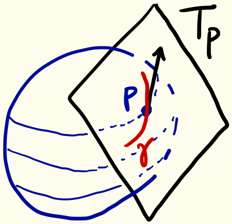
Where were we?

A C^∞ atlas is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that cover M and that are smothly sewn together.



A C^∞ n-dimensional manifold is a set M along with a maximal atlas, one that contains every possible compatible chart.

Tangent space



T_p = space of directional derivatives at p.

$$\boxed{\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\,\partial_\mu}$$

coordinate basis of T_p

Change of coordinates $\quad x^\mu \longrightarrow x^{\mu'}$

$\Rightarrow$ Change of basis $\quad \partial_{\mu'} = \dfrac{\partial x^\mu}{\partial x^{\mu'}}\,\partial_\mu$

$$\boxed{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}\,V^\nu}$$

Vector transformation law

$V = V^\mu \partial_\mu$

Exercise: Given two vector fields X and Y on M, show that:

- $X \circ Y$ is not a vector field

  Hint: act on a function $f : M \to \mathbb{R}$ $\quad X \circ Y(f) = X(Y(f))$

- $[X, Y]$ is a vector field with components

$$[X, Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu$$

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

Plan for the lecture

- Tensors

- Normal coordinates

- Differential forms

One-forms = dual vectors

Cotangent space $T_p^* = \{$ space of linear maps $\omega : T_p \longrightarrow \mathbb{R} \}$

$$\omega(V) = \omega(V^\mu \partial_\mu) = V^\mu \underbrace{\omega(\partial_\mu)}_{\equiv\, \omega_\mu} = V^\mu \omega_\mu$$

example: $\omega = df$ ← gradient of $f : M \to \mathbb{R}$

$$\underset{T_p^*}{\underset{\cap}{df}} \Big(\underset{T_p}{\underset{\cap}{\frac{d}{d\lambda}}} \Big) \equiv \frac{df}{d\lambda} \in \mathbb{R}$$

Notice that $\omega = \omega_\mu dx^\mu$ ↳ coordinate basis of T_p^*

$$\left[\begin{aligned} \text{Check: } \omega(V) &= \omega_\mu dx^\mu (V^\nu \partial_\nu) \\ &= \omega_\mu V^\nu \underbrace{dx^\mu(\partial_\nu)}_{\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu} = \omega_\mu V^\mu \end{aligned} \right]$$

Coordinate transformation

$x^\mu \to x^{\mu'}$ $\Longrightarrow$ $\boxed{\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu}$

[so that $\omega = \omega_\mu dx^\mu = \omega_{\mu'} dx^{\mu'}$]

A (k,ℓ) <u>Tensor</u> is a linear map $T: \underbrace{T_p \otimes \dots \otimes T_p}_{k} \otimes \underbrace{T_p^* \otimes \dots \otimes T_p^*}_{\ell} \longrightarrow \mathbb{R}$

basis of T_p

$$T = \underbrace{T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}}_{\substack{\text{components} \\ \text{in coordinate basis}}} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_\ell}$$

basis of T_p^*

Change of coordinates

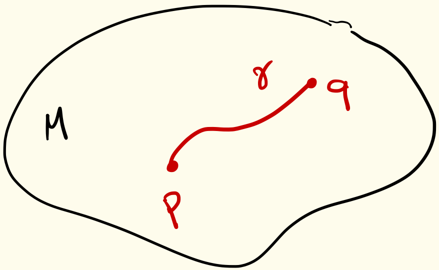
$$\boxed{T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_\ell'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_\ell}}{\partial x^{\nu_\ell'}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}}$$

The metric is a $(0,2)$ symmetric tensor that determines the line element:

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu$$

metric components

We can use the metric to determine the length of curves in a manifold.



$\gamma(0) = p$

$\gamma(1) = q$

$$\text{Length} = \int_0^1 d\lambda \sqrt{g_{\mu\nu}(x(\lambda)) \, \dot{x}^\mu \, \dot{x}^\nu} \, , \quad \bullet \equiv \frac{d}{d\lambda}$$

$$\text{Length} = \int_p^q ds$$

The metric is a (0,2) symmetric tensor that determines the line element:

$$ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu$$

metric components

- inverse metric : $g^{\alpha\beta}$, $g^{\alpha\beta} g_{\beta\mu} = g_{\mu\beta} g^{\beta\alpha} = \delta^\alpha_\mu$
- determinant : $g = \det g_{\alpha\beta} \neq 0$ <=> non-degenerate metric
- Canonical form : $g_{\alpha\beta} \to \mathrm{diag}(\underbrace{-1, \dots, -1}_{m}, \underbrace{1, \dots, 1}_{n})$
- Riemannian (or Euclidean) geometry $\Rightarrow$ $m = 0$
- Lorentzian " $\Rightarrow$ $m = 1$

Normal coordinates (at p)

$\exists$ coordinates $x^{\hat{\mu}}$ such that

$$\boxed{\begin{aligned} g_{\hat{\mu}\hat{\nu}}(p) &= \eta_{\hat{\mu}\hat{\nu}} \\ \partial_{\hat{\alpha}} g_{\hat{\mu}\hat{\nu}}(p) &= 0 \end{aligned}}$$

$$\partial_{\hat{\alpha}}\partial_{\hat{\beta}}\, g_{\hat{\mu}\hat{\nu}}(p) = \text{“curvature”}$$

$$g_{\hat{\mu}\hat{\nu}} = \underbrace{\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}}}_{\text{16 ind. comp.}} \; \underbrace{g_{\mu\nu}}_{\text{10 ind. components}}$$

Normal coordinates (at p)

$\exists$ coordinates $x^{\hat\mu}$ such that

$$\boxed{\begin{aligned} g_{\hat\mu\hat\nu}(p) &= \eta_{\hat\mu\hat\nu} \\ \partial_{\hat\alpha}\, g_{\hat\mu\hat\nu}(p) &= 0 \end{aligned}}$$

This fact is very useful!

Example: Observer with velocity U^μ and rocket with velocity V^μ.
What 3-velocity of the rocket does the observer see?

Normal coordinates

$$U^{\hat\mu} = (1,0,0,0)$$
$$V^{\hat\mu} = (\gamma, \gamma\vec{v})$$

$$\gamma = -\eta_{\hat\mu\hat\nu}\, U^{\hat\mu} V^{\hat\nu} = \frac{1}{\sqrt{1-|\vec{v}|^2}}$$

$$\Rightarrow \quad |\vec{v}| = \sqrt{1-\left(U_{\hat\mu} V^{\hat\mu}\right)^{-2}}$$

$\longrightarrow$

General coordinates

$$|\vec{v}|_U = \sqrt{1-\left(U_\mu V^\mu\right)^{-2}}$$

Check that this agrees with usual formulas for addition of velocities.

Differential Forms

A **p-form** is a $(0,p)$ tensor that is completely **anti-symmetric**.

n-dimensional manifold $\Rightarrow$ $\dfrac{n!}{p!\,(n-p)!}$ lin. ind. p-forms at any given point

$n=4$

$p =$	0	1	2	3	4
# I.C. =	1	4	6	4	1

Wedge product

$$\underbrace{\overset{p\text{-form}}{A} \wedge \overset{q\text{-form}}{B}}_{(p+q)\text{-form}}$$

$$\boxed{(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}}$$

Show that :

- $A \wedge B = B \wedge A \, (-1)^{pq}$
- $(A \wedge B)_{\mu\nu} = A_\mu B_\nu - B_\mu A_\nu$

1-forms

Exterior derivative

$$\underbrace{d\overset{\swarrow\ p\text{-form}}{A}}_{(p+1)\text{-form}}$$

$$\boxed{(dA)_{\mu_1\cdots\mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]}}$$

Examples: $p=0 \qquad (df)_\mu = \partial_\mu f$

$p=1 \qquad (dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Exercise: Show that dA transforms like a $(0,p+1)$ tensor under coordinate transformations.

$$(dA)_{\mu_1'\cdots\mu_{p+1}'} = (p+1)\frac{\partial}{\partial x^{[\mu_1'}}\left(\overbrace{\frac{\partial x^{\nu_2}}{\partial x^{\mu_2'}}\cdots\frac{\partial x^{\nu_{p+1}}}{\partial x^{\mu_{p+1}']}} A_{\nu_2\cdots\nu_{p+1}}}^{A_{\mu_2'\cdots\mu_{p+1}'}}\right)$$

$$= (p+1)\frac{\partial x^{\nu_2}}{\partial x^{[\mu_2'}}\cdots\frac{\partial x^{\nu_{p+1}}}{\partial x^{\mu_{p+1}'}}\underbrace{\frac{\partial}{\partial x^{\mu_1']}}}_{=\frac{\partial x^{\nu_1}}{\partial x^{\mu_1'}}\frac{\partial}{\partial x^{\nu_1}}} A_{\nu_2\cdots\nu_{p+1}} + \cdots \cancel{\frac{\partial^2 x^{\nu_2}}{\partial x^{\mu_1'}\partial x^{\mu_2'}}}\cdots = 0$$

anti-symmetrization $\Rightarrow 0$

Exterior derivative

$$\underbrace{d\overset{\swarrow \text{p-form}}{A}}_{(p+1)\text{-form}}$$

$$\boxed{(dA)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}}$$

Important property : $\boxed{d^2 = 0}$ $\qquad d(dA) = 0$

If $dA = 0$ (A is closed) then $A = dB$ (A is exact) locally, but not globally.

in an open region ($\sim \mathbb{R}^n$)

de Rham cohomology

$$H^p(M) = \frac{\text{closed p-forms on } M}{\text{exact p-forms on } M} \leftarrow \text{topological invariant}$$

Integration

Orientable n-dimensional manifold $\Leftrightarrow$ $\exists$ n-form continuous and nowhere vanishing

$$\alpha = h \; dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

$$\int_{U \subset M} \alpha \equiv \int_{\phi(U) \subset \mathbb{R}^n} h \, dx^1 \dots dx^n$$

coordinate chart

Exercise: show that $\int_U \alpha$ is coordinate invariant.

The volume form or Levi-Civita tensor

If the manifold is equipped with a metric, then there is a special n-form ϵ such that

signature $\begin{cases} s=0 & \text{Euc.} \\ s=1 & \text{Lor} \end{cases}$

$$\boxed{\epsilon_{\mu_1 \cdots \mu_n} \epsilon_{\nu_1 \cdots \nu_n} g^{\mu_1 \nu_1} \cdots g^{\mu_n \nu_n} = (-1)^s \, n!}$$

$$\Rightarrow \quad \epsilon = \sqrt{|g|} \, \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \equiv \sqrt{|g|} \, d^n x$$

Levi-Civita symbol $= \begin{cases} +1 & \text{if } (\mu_1 \cdots \mu_n) \text{ is an even perm. of } (0 1 \ldots n-1) \\ -1 & \text{" " " odd " "} \\ 0 & \text{otherwise} \end{cases}$

We can then define integrals over M:

$$\int_M \epsilon \, \psi(x) = \int_M \psi \sqrt{|g|} \, d^n x$$

function (0-form)